Level curvature distribution in a model of two uncoupled chaotic subsystems

Güler Ergün* and Yan V. Fyodorov†
Department of Mathematical Sciences, Brunel University, Uxbridge UB8 3PH, United Kingdom
(Received 18 March 2003; published 22 October 2003)

We study distributions of eigenvalue curvatures for a block-diagonal random matrix perturbed by a full random matrix. The most natural physical realization of this model is a quantum chaotic system with some inherent symmetry, such that its energy levels form two independent subsequences, subject to a generic perturbation which does not respect the symmetry. We describe analytically a crossover in the form of a curvature distribution with a tunable parameter, namely, the ratio of intersubsystem/intrasubsystem coupling strengths. We find that the peak value of the curvature distribution is much more sensitive to the changes in this parameter than the power-law tail behavior. This observation may help to clarify some qualitative features of the curvature distributions observed experimentally in acoustic resonances of quartz blocks.

DOI: 10.1103/PhysRevE.68.046124 PACS number(s): 02.50.−r, 03.65.−w, 05.45.Pq

I. INTRODUCTION

Studies of statistical properties of complex quantum systems (chaotic or disordered) show that their eigenvalue spectra exhibit patterns of universal fluctuations, whose structure mainly depends on the fundamental symmetries of the Hamiltonian [1,2]. Such a universality opens an attractive possibility of modeling the fluctuations by comparing them with those observed in long sequences of eigenvalues of random matrices of appropriate symmetry [3,4]. Namely, systems with no time-reversal invariance are known to be adequately described by Gaussian unitary ensemble (GUE) of complex Hermitian matrices, and systems with time-reversal invariance are described by Gaussian orthogonal ensemble-(GOE) or Gaussian symplectic ensemble of real symmetric or complex quaternion matrices, depending on the existence of strong spin-orbit coupling.

More recently, the interest in studying spectral statistics of quantum chaotic systems was much revitalized by the understanding that their energy spectra display a universal change in their characteristics as a response to external perturbations of various kinds. The nature of the perturbation may vary considerably depending on the physical system, and usually involves application of external fields (magnetic or electric), change of boundary conditions or the shape of the system, rearrangement of positions of impurities in disordered medium, or variation of temperature, pressure or any other tunable physical characteristics.

One of the frequently used measures of parametric sensitivities of complex quantum systems is the distribution of level curvatures, which is defined as the second-order derivative of eigenvalues with respect to a perturbation parameter.

In Ref. [5] Gaspard and coauthors developed expressions for the probability densities of level curvatures and found that the large curvatures must exhibit universal behavior classified according to the underlying gross symmetries. Indeed, the curvatures become large in the vicinity of avoided crossings of energy levels as functions of a parameter and their distribution can be simply related to that of small eigenvalue spacings. On the basis of extensive numerical investigations of both random matrices and several quantum chaotic systems, Zakrzewski and Delande conjectured an analytical expression for the full distribution of curvatures [6]. Later on, Zakrzewski-Delande formulas were derived analytically by von Oppen [7] and by Fyodorov and Sommers [8] for the random matrix models of all three universality classes.

However, in many relevant experimental circumstances, physical systems have accidentally more underlying symmetries (frequently called ''geometric'') acting in addition to the presence or the absence of the time-reversal invariance. Such symmetries naturally induce classification of energy levels according to irreducible representations of the corresponding symmetry group, and the energy levels corresponding to different representations form statistically independent subsequences. Only these subsequences may be then meaningfully compared with the universal random matrix patterns. In fact, a generic situation may be even more complicated, since geometric symmetries may not be exact, but approximate. This will clearly lead to spectra being a mixture of different subsequences with uncertain statistical consequences of mutual interference.

Recent experimental studies on acoustic resonance spectra in quartz blocks [9,10] suggest that the system may fall into the latter category, and the deviations from standard theoretical predictions of parametric correlations may have their origin in remnant geometric symmetries. This fact motivated several groups to investigate the effects of partial symmetry breaking on the level curvatures [11,12]. However, closed form analytical expressions for curvature distributions for the case of partly broken symmetries are not available, to the best of our knowledge.

In the present paper we consider a simpler, but related model. Rather than studying level curvatures in a system with partly broken symmetry, we address the case of two noninteracting subsystems subject to a perturbation which induces both coupling between the subsystems and variation of parameters within each of the two subsystems. This should be generically the case for a perturbation which does not respect the underlying symmetry. Of course, we do not

*Electronic address: Guler.Ergun@brunel.ac.uk
†Electronic address: Yan.Fyodorov@brunel.ac.uk

claim our simple model to be adequate for describing experimental situation in quartz blocks, but we rather hope that our results could help in indicating how various factors may affect the shape of level curvature distributions.

Employing the random matrix calculations allows one to derive exact expressions for the level curvature distribution as a function of relative weight of induced intrasublattice and intersublattice osculating variations. The curvature distribution naturally interpolates between the simple Cauchy-Lorentz shape—characteristic for pure symmetry-breaking perturbations which couple two subsystems without modifying them individually—and the Zakrzewski-Delande formulas typical for perturbations respecting the underlying symmetry.

We first treat a simpler case of complex Hermitian matrices (systems with broken time-reversal invariance) in detail, and then extend the derivations to the case of real symmetric matrices. Our analytical calculations are supported and corroborated by accurate numerical simulations of random matrix ensembles.

II. GENERAL RELATIONS

To study the parametric dependence of energy levels of a system with some underlying symmetry, we consider a random matrix model where the Hamiltonian $\mathcal{H}$ of the system linearly depends on a perturbation parameter ε:

$$\mathcal{H}(\varepsilon)=\hat{A}+\varepsilon\hat{B}. \tag{1}$$

For the unperturbed Hamiltonian $\hat{A}$ we choose a block-diagonal matrix

$$\hat{A}=\begin{pmatrix}\hat{H}_1 & \oslash\\ \oslash & \hat{H}_2\end{pmatrix},$$

with $\hat{H}_{1,2}$ being $N\times N$ random matrices (complex Hermitian or real symmetric) taken either from GUE or, respectively, from GOE. These can be thought of as representing two noninteracting chaotic subsystems. That means, we consider the matrices $\hat{H}_p$ as random Gaussian, with entries $H_{i>j}$ being independent and identically distributed variables with mean zero and variances $\langle \overline{H_{ij}}H_{ij}\rangle=2\sigma^2/N$ for the GUE case ($\beta=2$) and $\langle H_{ij}^2\rangle=\sigma^2/2N$ for GOE case ($\beta=1$). The joint probability density for $\hat{H}_p$ is then written as

$$\mathcal{P}(\hat{H}_p)=C_N^\beta \exp\left\{-\frac{N}{2\beta\sigma^2}\,\mathrm{Tr}\,\hat{H}_p^2\right\}, \tag{2}$$

where C_N^β is the appropriate normalization constant.

We introduce an interaction between blocks $\hat{H}_{1,2}$ by considering a general coupling matrix

$$\hat{B}=\begin{pmatrix}\hat{J}_1 & \hat{V}\\ \hat{V}^\dagger & \hat{J}_2\end{pmatrix}.$$

We assume here that $\hat{J}_{1,2}$ have the same symmetry properties as the diagonal blocks $\hat{H}_{1,2}$ and their probability densities can be written as

$$\mathcal{P}(\hat{J}_p)\propto\exp\left\{-\frac{N}{2\beta\sigma_1^2}\,\mathrm{Tr}\,\hat{J}_p^2\right\}. \tag{3}$$

As for the off-diagonal blocks $\hat{V}$, they represent general Gaussian random matrices (complex for $\beta=2$ or real for $\beta=1$) with no further symmetry constraints imposed. The probability density for $\hat{V}$ is then chosen for both cases to be

$$\mathcal{P}(\hat{V})\propto\exp\left\{-\frac{N}{2\sigma_2^2}\,\mathrm{Tr}\,\hat{V}^\dagger\hat{V}\right\}. \tag{4}$$

Models of this kind were previously employed with satisfactory results in the analysis of data relative to symmetry breaking in nuclear physics [14].

We denote the eigenvalues and corresponding eigenvectors of $\hat{H}_p$ by $[\lambda_i^{(p)},\mathbf{v}_i^{(p)}]$ with $p=1,2$. This implies that $\hat{H}_p\mathbf{v}_i^{(p)}=\lambda_i^{(p)}\mathbf{v}_i^{(p)}$ where $i=1,\ldots,N$ and $\mathbf{v}_i^{(p)\dagger}\mathbf{v}_i^{(p)}=1$. Our main goal is to find the distribution of level curvatures, defined as the second-order derivative of eigenvalues of $\mathcal{H}$ with respect to the perturbation parameter ε. Employing in the usual way the second-order perturbation theory we can write the expression for the curvature corresponding, say, to an eigenvalue that for $\varepsilon=0$ coincides with the (unperturbed) eigenvalue $\lambda_i^{(1)}$ as

$$\mathcal{C}_i=\sum_{k\neq i}^{N}\frac{(\mathbf{v}_{k1}^\dagger\hat{J}_1\mathbf{v}_{i1})(\mathbf{v}_{i1}^\dagger\hat{J}_1^\dagger\mathbf{v}_{k1})}{\lambda_i^{(1)}-\lambda_k^{(1)}}+\sum_{k=1}^{N}\frac{(\mathbf{v}_{i1}^\dagger\hat{V}\mathbf{v}_{k2})(\mathbf{v}_{k2}^\dagger\hat{V}^\dagger\mathbf{v}_{i1})}{\lambda_i^{(1)}-\lambda_k^{(2)}}. \tag{5}$$

This expression shows that there are generically two contributions to the level curvatures. The first sum is essentially the level curvature induced by the *block-diagonal* part of the perturbation which does not lead to any mixing between levels of the two noninteracting subsystems. Taken alone, this term (which we will denote here as $\mathcal{C}_{1i}$) must, therefore, yield the Zakrzewski-Delande curvature distribution. In contrast, the second sum that will be denoted as $\mathcal{C}_{2i}$ reflects the influence of the off-diagonal perturbation, which mixes the levels of two subsystems.

Then the distribution of the total curvatures is defined as $\mathcal{P}(\mathcal{C})=\langle\delta(\mathcal{C}-\mathcal{C}_{1i}-\mathcal{C}_{2i})\rangle_{\mathcal{H}}$ and it can be conveniently written using the Fourier transform as

$$\mathcal{P}(\mathcal{C})=\frac{1}{2\pi}\int_{-\infty}^{\infty}dx e^{ix\mathcal{C}}\langle\langle e^{-ix\mathcal{C}_{1i}}\rangle_{\hat{J}_1}\langle e^{-ix\mathcal{C}_{2i}}\rangle_{\hat{V}}\rangle_{\hat{H}_1,\hat{H}_2}. \tag{6}$$

The first factor $\langle\langle e^{-ix\mathcal{C}_{1i}}\rangle_{\hat{J}_1}\rangle_{\hat{H}_1}$ is just the Fourier transform of the known Zakrzewski-Delande expression, hence the calculation of the curvature distribution reduces to evaluating the remaining factor $\langle\langle e^{-ix\mathcal{C}_{2i}}\rangle_{\hat{V}}\rangle_{\hat{H}_2}$. Our next goal is to derive the corresponding expressions, first for $\beta=2$ and then for $\beta=1$.

III. COMPLEX HERMITIAN MATRICES: $\beta=2$

We are going to evaluate the following ensemble average [see Eq. (5)]:

$$\left\langle \exp\left(-ix\sum_{k=1}^{N} \frac{1}{\lambda_i^{(1)}-\lambda_k^{(2)}}(\mathbf{v}_{i1}^{\dagger}\hat{V}\mathbf{v}_{k2})(\mathbf{v}_{k2}^{\dagger}\hat{V}^{\dagger}\mathbf{v}_{i1})\right)\right\rangle_{\hat{V},\hat{H}_2}. \tag{7}$$

First we perform the average over $\hat{V}$. To simplify the notation we denote $x/(\lambda_i^{(1)}-\lambda_k^{(2)})\equiv b_k$, $\mathbf{v}_{i1}^{\dagger}\hat{V}\mathbf{v}_{k2}\equiv w_k$ and use the identity

$$e^{-ib_k\bar{w}_k w_k}=-\frac{i}{b_k}\int\frac{d\bar{Z}_k dZ_k}{2\pi}e^{(i/b_k)\bar{Z}_k Z_k - i(Z_k w_k+\bar{Z}_k\bar{w}_k)}, \tag{8}$$

where the integration is taken over an auxiliary complex variable Z_k. This allows one to rewrite Eq. (7) in the following form:

$$\left\langle \prod_{k=1}^{N}\left(-\frac{i}{b_k}\right)\int\frac{d\bar{Z}_k dZ_k}{2\pi}e^{(i/b_k)\bar{Z}_k Z_k - i(Z_k w_k+\bar{Z}_k\bar{w}_k)}\right\rangle_{\hat{V},\hat{H}_2}. \tag{9}$$

Explicitly, employing the distribution function in Eq. (4), we need to calculate the integral

$$\int d\hat{V}d\hat{V}^{\dagger}\exp\left\{-\frac{N}{2\sigma_2^2}\mathrm{Tr}(\hat{V}\hat{V}^{\dagger})-i\mathbf{v}_{i1}^{\dagger}\hat{V}\left(\sum_{k=1}^{N}Z_k\mathbf{v}_{k2}\right)-i\left(\sum_{k=1}^{N}\bar{Z}_k\mathbf{v}_{k2}^{\dagger}\right)\hat{V}^{\dagger}\mathbf{v}_{i1}\right\}. \tag{10}$$

It is convenient to introduce an $N\times N$ matrix $\hat{G}=(\Sigma_k Z_k\mathbf{v}_{k2})\otimes\mathbf{v}_{i1}^{\dagger}$, so that, when used in the former identity relation, we get

$$\int d\hat{V}d\hat{V}^{\dagger}e^{-(N/2\sigma_2^2)\mathrm{Tr}(\hat{V}\hat{V}^{\dagger})-i\mathrm{Tr}(\hat{V}\hat{G}+\hat{G}^{\dagger}\hat{V}^{\dagger})}\propto e^{-(2\sigma_2^2/N)\mathrm{Tr}(\hat{G}^{\dagger}\hat{G})}, \tag{11}$$

which is a generalization of Eq. (8). We further use

$$\mathrm{Tr}(\hat{G}^{\dagger}\hat{G})=(\mathbf{v}_{i1}^{\dagger}\mathbf{v}_{i1})\sum_{k\varrho}\bar{Z}_k Z_{\varrho}(\mathbf{v}_{k2}^{\dagger}\mathbf{v}_{\varrho 2}), \tag{12}$$

and recall that $\mathbf{v}_{i1}$ are eigenvectors of $\hat{H}_1$ and $\mathbf{v}_{k2}$ are those of $\hat{H}_2$. Using the orthogonality of the eigenvectors,

$$\mathbf{v}_{k2}^{\dagger}\mathbf{v}_{\varrho 2}=\begin{cases}1, & k=\varrho\\ 0, & k\neq\varrho\end{cases}$$

and $\mathbf{v}_{i1}^{\dagger}\mathbf{v}_{i1}=1$ gives the result for the average in Eq. (9) as

$$\left\langle\prod_{k=1}^{N}\int\frac{d\bar{Z}_k dZ_k}{2\pi i b_k}e^{-2\sigma_2^2/N\bar{Z}_k Z_k+(i/b_k)\bar{Z}_k Z_k}\right\rangle_{\hat{H}_2}\propto\left\langle\frac{\det(\lambda_i^{(1)}I_N-\hat{H}_2)}{\det\left[\left(\lambda_i^{(1)}+\frac{i\tilde{x}}{N}\right)I_N-\hat{H}_2\right]}\right\rangle_{\hat{H}_2}. \tag{13}$$

For this we also performed the Gaussian integrations over Z_k explicitly, and denoted $\tilde{x}=2\sigma_2^2 x$.

In what follows, we denote $\lambda_i^{(1)}\equiv\lambda$ and $\lambda_i^{(1)}+i\tilde{x}/N\equiv\lambda_b$ and proceed with calculations of the average $\langle\cdots\rangle_{\hat{H}_2}$ in Eq. (13) by employing a technique suggested in Ref. [13]. In fact, for $\beta=2$ the averages of the ratios of determinants are known in full generality for any value of N [13,15]. Nevertheless, we outline the corresponding calculation in order to introduce the method and the convenient notation which will be used later on in this paper for the more complicated case $\beta=1$.

Using the standard ''supersymmetrization'' idea [16], we represent the denominator of the expression to be averaged as a Gaussian integral (we assume here $x>0$ for definiteness)

$$\det{}^{-1}(\lambda_b I_N-\hat{H}_2)=\frac{1}{i^N}\int d\mathbf{S}d\mathbf{S}^{\dagger}e^{(i/2)\lambda_b\mathbf{S}^{\dagger}\mathbf{S}-(i/2)\mathbf{S}^{\dagger}\hat{H}_2\mathbf{S}}, \tag{14}$$

with a complex N-dimensional vector $\mathbf{S}=(S_1,\ldots,S_N)^T$, where T stands for the vector transpose. For the determinant in the numerator, we use Gaussian integrals over anticommuting (Grassmannian) N-component vectors $\chi,\chi^{\dagger}$, which gives

$$\det(\lambda I_N-\hat{H}_2)=\frac{1}{i^N}\int d\chi d\chi^{\dagger}e^{(i/2)\lambda\chi^{\dagger}\chi-(i/2)\chi^{\dagger}\hat{H}_2\chi}. \tag{15}$$

Substituting the relations $\chi^{\dagger}\hat{H}_2\chi=-\mathrm{Tr}(\hat{H}_2\chi\otimes\chi^{\dagger})$ and $\mathbf{S}^{\dagger}\hat{H}_2\mathbf{S}=\mathrm{Tr}(\hat{H}_2\mathbf{S}\otimes\mathbf{S}^{\dagger})$ in the integral yields

$$\langle\cdots\rangle_{\hat{H}_2}=\int d^2\chi\int d^2\mathbf{S}e^{(i/2)[\lambda\chi^{\dagger}\chi+\lambda_b\mathbf{S}^{\dagger}\mathbf{S}]}\times\langle e^{-(i/2)\mathrm{Tr}\hat{H}_2[\mathbf{S}\otimes\mathbf{S}^{\dagger}-\chi\otimes\chi^{\dagger}]}\rangle_{\hat{H}_2}. \tag{16}$$

The ensemble average over GUE matrices $\hat{H}_2$ can be easily performed by exploiting the identity

$$\langle e^{-(i/2)\mathrm{Tr}[\hat{H}\hat{A}]}\rangle_{GUE}\propto e^{-(\sigma^2/4N)\mathrm{Tr}[\hat{A}^2]} \tag{17}$$

and ''decoupling'' the quartic term in Grassmann variables with the help of simple Hubbard-Stratonovich transformation:

$$e^{(\sigma^2/4N)(\chi^{\dagger}\chi)^2}=\int_{-\infty}^{\infty}\frac{dq}{\sqrt{2\pi}}e^{-(q^2/2)-(q\sigma/\sqrt{2N})\chi^{\dagger}\chi}. \tag{18}$$

After some straightforward manipulations we arrive at the following integral representation for the required ensemble average:

$$\langle \cdots \rangle_{\hat{H}_2} = \int d^2\mathbf{S} e^{-(\sigma^2/4N)(\mathbf{S}^\dagger\mathbf{S})^2 + (i/2)\lambda_b \mathbf{S}^\dagger\mathbf{S}} \times \int_{-\infty}^{\infty} \frac{dq}{\sqrt{2\pi}} e^{-(q^2/2)} \times \det\left[\frac{1}{2}\left(i\lambda - \frac{q\sigma}{N/2}\right)\hat{I}_N - \frac{\mathbf{S}\otimes\mathbf{S}^\dagger}{2N/\sigma^2}\right]. \quad (19)$$

Introducing the variable $q_F = i\lambda - q\sigma/\sqrt{N/2}$ and shifting the contour of integration in such a way that, the integral over q_F goes along the real axis, we can rewrite the above expression as

$$\langle \cdots \rangle_{\hat{H}_2} \propto \int_{-\infty}^{\infty} \frac{dq_F}{\sqrt{2\pi}} e^{-(N/4\sigma^2)(q_F - i\lambda)^2} \times \int d^2\mathbf{S} e^{(\sigma^2/4N)(\mathbf{S}^\dagger\mathbf{S})^2 + (i/2)\lambda_b \mathbf{S}^\dagger\mathbf{S}} \times \det\left[q_F\hat{I} - \frac{\sigma^2}{N}\mathbf{S}\otimes\mathbf{S}^\dagger\right], \quad (20)$$

where we shifted the contour for $q_F \in (-\infty,\infty)$ to be real. Further simplification can be made by noticing that the $N\times N$ matrix $\mathbf{S}\otimes\mathbf{S}^\dagger$ is of rank unity, i.e., it has $(N-1)$ zero eigenvalues, and only one nonzero eigenvalue equal to $(\mathbf{S}\mathbf{S}^\dagger)$. Then the determinant in the previous expression is

$$\det\left[q_F\hat{I} - \frac{\sigma^2}{N}\mathbf{S}\otimes\mathbf{S}^\dagger\right] \equiv q_F^{N-1}\left(q_F - \frac{\sigma^2}{N}\mathbf{S}\mathbf{S}^\dagger\right). \quad (21)$$

Finally, we introduce polar coordinates $\mathbf{S} = r\mathbf{n}$ with $\mathbf{n}^\dagger\mathbf{n} = 1$ and $\int d^2\mathbf{S} = r^{2N-1} dr\, d\mathbf{n}$, where $\int d\mathbf{n} = \Omega_N$ produces a constant factor, which corresponds to the area of a $2N$-dimensional unit sphere. Further introducing $p = r^2$ and changing $p \to Np/\sigma^2$ and then following with the obvious manipulations we get

$$\left\langle \frac{\det(\lambda\hat{I}_N - \hat{H})}{\det(\lambda_b\hat{I}_N - \hat{H})} \right\rangle_{\hat{H}_2} = C_N e^{(N/4\sigma^2)\lambda^2} \int_{-\infty}^{\infty} \frac{dq_F}{\sqrt{2\pi} q_F} \times e^{-(N/4\sigma^2)(q_F^2 - 2i\lambda q_F - 4\sigma^2 \ln q_F)} \times \int_0^{\infty} \frac{dp(q_F - p)}{\sqrt{2\pi p}} e^{-x(\sigma_2^2/\sigma^2)p} \times e^{-(N/4\sigma^2)(p^2 - 2i\lambda p - 4\sigma^2 \ln p)}, \quad (22)$$

where we reinstated $\lambda_b = \lambda + i\tilde{x}/N$, $\tilde{x} = 2\sigma^2 x$ and C_N stands for the accumulated constant factors. The latter can always be restored by noticing that when $\lambda_b = \lambda$ the right hand side must yield unity identically.

So far all the expressions were valid for finite-size matrices. When $N\to\infty$ we expect the results, when appropriately scaled, to be universal, i.e., broadly insensitive to the details of the distribution of random matrices and applicable to quantum chaotic systems. In such a limit the integrals in Eq. (22) can be evaluated by the saddle-point method. For $\lambda \leqslant \sqrt{8\sigma^2}$ (the so called *bulk* of the spectrum) the relevant saddle points (sp) are $p^{sp} = (i\lambda + \sqrt{8\sigma^2 - \lambda^2})/2$ and $q_F^{sp} = (i\lambda \pm \sqrt{8\sigma^2 - \lambda^2})/2$. It is easy to see that only the choice $q_F^{sp} = (i\lambda - \sqrt{8\sigma^2 - \lambda^2})/2$ yields the leading-order contribution, due to the presence of the factor $(q_F - p)$ in the integrand. Substituting this choice into the integrand in Eq. (22) and evaluating the Gaussian fluctuations around the saddle-point values finally yields

$$\left\langle \frac{\det(\lambda\hat{I}_N - \hat{H})}{\det\left[\left(\lambda + i\frac{\tilde{x}}{N}\right)\hat{I}_N - \hat{H}\right]} \right\rangle \Bigg|_{\substack{x>0 \\ N\to\infty}} = \exp\left\{ -\frac{x}{2}[i\lambda + \sqrt{8\sigma^2 - \lambda^2}]\left(\frac{\sigma_2}{\sigma}\right)^2 \right\}. \quad (23)$$

It is easy to repeat the calculation for $x<0$ and find that for any real value of x the result can be written as

$$\left\langle \frac{\det(\lambda\hat{I}_N - \hat{H})}{\det\left[\left(\lambda + i\frac{\tilde{x}}{N}\right)\hat{I}_N - \hat{H}\right]} \right\rangle \Bigg|_{N\to\infty} = \exp\left\{ -i\lambda x\frac{\sigma_2^2}{2\sigma^2} - 2|x|\pi\rho(\lambda)\sigma_2^2 \right\}, \quad (24)$$

where $\rho(\lambda) = (1/4\pi\sigma^2)\sqrt{8\sigma^2 - \lambda^2}$ is the mean eigenvalue density for GUE.

The Fourier transform of the above expression with respect to x immediately gives us the distribution $\mathcal{P}_{off}(\mathcal{C})$ of level curvatures induced by purely off-diagonal random coupling $\hat{V}$ between the two subsystems. In the large-size limit $N\to\infty$ we therefore have

$$\mathcal{P}_{off}(\mathcal{C}) = \frac{1}{\pi} \frac{2\sigma_2^2\pi\rho(\lambda)}{\left(\mathcal{C} - \lambda\frac{\sigma_2^2}{2\sigma^2}\right)^2 + (2\pi\rho(\lambda)\sigma_2^2)^2}, \quad (25)$$

which is nothing else but the Cauchy-Lorentz distribution with the mean value $\langle\mathcal{C}\rangle_{off} = \lambda(\sigma_2^2/2\sigma^2)$ and characteristic widths $\Gamma_{off} = [2\pi\rho(\lambda)\sigma_2^2]$.

Turning our attention to the curvatures induced by the block-diagonal contributions $\hat{J}_p$ [the term $\mathcal{C}_{1i}$ in Eq. (5)] we can first perform the ensemble average over the Gaussian distribution of $\hat{J}_1$, Eq. (3). Employing similar methods as before, we easily find the result to be

$$\langle e^{-ix\mathcal{C}_{1i}}\rangle_{\hat{J}_1}=\prod_{k\neq i}\frac{(\lambda_i^{(1)}-\lambda_k^{(1)})}{\left[\lambda_i^{(1)}-\frac{ix_1}{N}-\lambda_k^{(1)}\right]}, \tag{26}$$

where in this expression $x_1=2\sigma_1^2 x$. This expression remains to be averaged over the joint probability density of $N-1$ GUE eigenvalues $\lambda_1^{(1)},\ldots,\lambda_{i-1}^{(1)},\lambda_{i+1}^{(1)},\ldots,\lambda_N^{(1)}$ which are different from the chosen eigenvalue $\lambda_i^{(1)}$ whose curvature we address. The consideration which is exposed in Refs. [7,8] shows that

$$\langle e^{-ix\mathcal{C}_{1i}}\rangle_{\hat{J}_1,\hat{H}_1}\propto e^{-(N/4\sigma^2)\lambda^2}\left\langle\frac{\det^3(\lambda-H)}{\det\left(\lambda+\frac{ix_1}{N}-H\right)}\right\rangle_{N-1}, \tag{27}$$

where H is a $(N-1)\times(N-1)$ GUE matrix.

The averaging of the ratios of determinants in Eq. (27) can be done, *mutatis mutandis*, by the same supersymmetrization procedure as above. The detailed exposition of the corresponding calculation can be found in the paper by Fyodorov and Strahov [13]. Here we briefly sketch the main steps. After representing each of the four determinants by the Gaussian integrals (three over anticommuting and one over usual complex variables) one can easily perform the GUE average by exploiting the identity Eq. (17). Then the terms in the exponent quartic with respect to anticommuting variables are ''decoupled'' by introducing an auxiliary integration over 3×3 Hermitian matrix $\hat{Q}$, the procedure being a straightforward generalization of the Hubbard-Stratonovich transformation (18). All the subsequent manipulations are quite analogous to those exposed above, and for our case, instead of Eq. (22) we arrive at its analog pertinent:

$$\left\langle\frac{\det^3(\lambda-\tilde{H})}{\det(\lambda_b-\tilde{H})}\right\rangle_{\tilde{H}}\propto\int dQ_F(\det Q_F)^{\tilde{N}-1}\exp\left\{-\frac{\tilde{N}}{4\sigma^2}\mathrm{Tr}(Q_F-i\lambda\hat{I})^2\right\}\int_0^\infty dp\,p^{\tilde{N}-1}e^{-x(\sigma_2^2/\sigma^2)p}\exp\left\{-\frac{\tilde{N}}{4\sigma^2}(p^2-2i\lambda\,p)\right\}$$
$$\times(q_F^{(1)}-p)(q_F^{(2)}-p)(q_F^{(3)}-p), \tag{28}$$

where $q_F^{(1,2,3)}$ are real eigenvalues of the Hermitian 3×3 matrix Q_F and $\tilde{N}$ stands for $N-1$. In fact, since $\mathrm{Tr}(Q_F-i\lambda\hat{I})^2$ and $\det Q_F$ depend only on the eigenvalues $q_F^{(1,2,3)}$, it is convenient to use these eigenvalues and the corresponding eigenvectors as integration variables. In these coordinates the integration measure is given by

$$dQ_F\propto d\mu[U]dq_F^{(1)}dq_F^{(2)}dq_F^{(3)}\prod_{1\leqslant k_1<k_2\leqslant3}(q_F^{(k_1)}-q_F^{(k_2)}),$$

where $d\mu[U]$ is the invariant measure on the manifold of unitary 3×3 matrices, representing the eigenvectors of Q_F and the last factor is the Jacobian of the transformation, known as the Vandermonde determinant.

Again, we are interested in the limit $N\gg1$, so we neglect the difference between N and $N-1$ and omit the tilde henceforth. The set of the saddle points of the integrand with respect to each of the variables $p>0$ and $q_F^{(1,2,3)}$ is $p^{sp}=(i\lambda+\sqrt{8\sigma^2-\lambda^2}/2)$ and $q_F^{sp}=(i\lambda\pm\sqrt{8\sigma^2-\lambda^2}/2)$. These saddle points are the same as that we found earlier. However, the presence of both the Vandermonde factors and that of the product $\Pi_{k=1}^3(q_F^{(k)}-p)$ makes us select the following saddle points:

$$q_F^{(1)}=\frac{i\lambda+\sqrt{8\sigma^2-\lambda^2}}{2},\quad q_F^{(2)}=q_F^{(3)}=\frac{i\lambda-\sqrt{8\sigma^2-\lambda^2}}{2}$$

(as well as its cyclic permutations) as these give the leading-order contribution. In fact, the integrand vanishes at these saddle-point values and care should be taken to expand the integrand further when calculating the contribution from the Gaussian fluctuations around the saddle points (see Ref. [13] for a general procedure). The final result is given by

$$\langle e^{-ix\mathcal{C}_{1i}}\rangle_{\hat{J}_1,\hat{H}_1}=[1+2\pi\rho(\lambda)\sigma_1^2|x|]\times\exp\left\{-i\lambda x\frac{\sigma_1^2}{2\sigma^2}-2|x|\pi\rho(\lambda)\sigma_1^2\right\}. \tag{29}$$

Taking the Fourier transform, we, as expected, arrive at the Zakrzewski-Delande formula for $\beta=2$:

$$\mathcal{P}_{diag}(\mathcal{C})=\frac{2}{\pi}\frac{\Gamma_d^3}{[(\mathcal{C}-\langle\mathcal{C}\rangle_d)^2+\Gamma_d^2]^2}, \tag{30}$$

where the mean value $\langle\mathcal{C}\rangle_d=\lambda(\sigma_1^2/2\sigma^2)$ and the characteristic widths $\Gamma_d=[2\pi\rho(\lambda)\sigma_1^2]$.

Now we know all the factors in Eq. (6) and can find the curvature distribution accounting for both diagonal and off-diagonal perturbations of the two decoupled subsystems:

$$\mathcal{P}(\mathcal{C})=\int_{-\infty}^{\infty}\frac{dx}{2\pi}e^{ix\mathcal{C}}[1+\Gamma_d|x|]\times\exp\left\{-ix\frac{\lambda}{2\sigma^2}(\sigma_1^2+\sigma_2^2)-|x|(\Gamma_{off}+\Gamma_d)\right\}. \tag{31}$$

Performing the integration explicitly, we arrive at our final formula for complex Hermitian case:

$$\mathcal{P}(\mathcal{C})=\frac{1}{\pi}\left\{\frac{\Gamma_{off}}{[\mathcal{C}-\frac{1}{2}(\langle\mathcal{C}\rangle_{off}+\langle\mathcal{C}\rangle_{d})]^2+(\Gamma_{off}+\Gamma_{d})^2}+\frac{2\Gamma_{d}(\Gamma_{off}+\Gamma_{d})^2}{\{[\mathcal{C}-\frac{1}{2}(\langle\mathcal{C}\rangle_{off}+\langle\mathcal{C}\rangle_{d})]^2+(\Gamma_{off}+\Gamma_{d})^2\}^2}\right\}. \tag{32}$$

IV. REAL SYMMETRIC MATRICES: $\beta=1$

We again need to evaluate the ensemble average as in Eq. (7), but this time for the real-valued perturbation $\hat{V}$ and real-valued eigenvectors $\mathbf{v}_i$, so that the quantity $\mathbf{v}_{i1}^T\hat{V}\mathbf{v}_{k2}\equiv w_k$ is a real variable. As before we denote $x/(\lambda_i^{(1)}-\lambda_k^{(2)})\equiv b_k$ and use the integration over an auxiliary real variable x_k:

$$e^{-ib_kw_k^2}=-\sqrt{\frac{i}{b_k}}\int\frac{dx_k}{2\pi}e^{(i/4b_k)x_k^2-ix_kw_k}, \tag{33}$$

combined with the fact that [cf. Eq. (11)]

$$\int d\hat{V}\exp\left\{-\frac{N}{2\sigma_2^2}\mathrm{Tr}(\hat{V}\hat{V}^T)-i\,\mathrm{Tr}\hat{V}\sum_{k=1}^{N}x_k(\mathbf{v}_{k2}\otimes\mathbf{v}_{i1}^T)\right\}\propto\exp\left\{-\frac{\sigma_2^2}{2N}\sum_{k=1}^{N}x_k^2\right\} \tag{34}$$

because of the orthogonality of eigenvectors. Consequently, we easily perform the Gaussian integral over x_k and obtain

$$\langle\langle e^{-ix\mathcal{C}_{2i}}\rangle_V\rangle_{H_2}\propto\left\langle\frac{\det^{1/2}(\lambda_i^{(1)}I_N-\hat{H}_2)}{\det^{1/2}\left[\left(\lambda_i^{(1)}+\frac{i\tilde{x}}{N}\right)I_N-\hat{H}_2\right]}\right\rangle_{\hat{H}_2}, \tag{35}$$

as before we denoted $\tilde{x}=2\sigma_2^2x$.

After denoting $\lambda_i^{(1)}\equiv\lambda$ and $\lambda_i^{(1)}+i\tilde{x}/N\equiv\lambda_b$ for a less cumbersome expression, we then proceed with calculations of the average $\langle\cdots\rangle_{\hat{H}_2}$ in Eq. (35). To be able to employ the previous technique for $\beta=2$ case we first rewrite

$$\frac{\det^{1/2}(\lambda I_N-\hat{H}_2)}{\det^{1/2}(\lambda_bI_N-\hat{H}_2)}\equiv\frac{\det(\lambda I_N-\hat{H}_2)}{\det^{1/2}(\lambda_bI_N-\hat{H}_2)\det^{1/2}(\lambda I_N-\hat{H}_2)}. \tag{36}$$

Assuming, for definiteness, $\tilde{x}<0$, and also assuming that λ has an infinitesimal negative imaginary part we can represent the two factors in the denominator as Gaussian integrals over real N component vectors $\mathbf{x}_{1,2}$:

$$\det{}^{-1/2}(\lambda I_N-\hat{H}_2)\propto\int d\mathbf{x}_1e^{-(i/2)\lambda\mathbf{x}_1^T\mathbf{x}_1+(i/2)\mathbf{x}_1^T\hat{H}_2\mathbf{x}_1} \tag{37}$$

and

$$\det{}^{-1/2}(\lambda_bI_N-\hat{H}_2)\propto\int d\mathbf{x}_2e^{-(i/2)\lambda_b\mathbf{x}_2^T\mathbf{x}_2+(i/2)\mathbf{x}_2^T\hat{H}_2\mathbf{x}_2}, \tag{38}$$

where T stands for vector transpose. As for the determinant in the numerator, we can use the same Gaussian integral Eq. (15) over anticommuting (Grassmannian) N-component vectors $\chi,\chi^\dagger$. Substituting these integral representations to Eq. (36) and performing the ensemble averaging over GOE matrix $\hat{H}_2$, with the help of identity

$$\langle e^{\pm i/2\mathrm{Tr}[\hat{H}\hat{A}]}\rangle_{GOE}\propto e^{-(\sigma^2/32N)\mathrm{Tr}(\hat{A}^T+\hat{A})^2}, \tag{39}$$

one can satisfy oneself that the resulting expression takes the form

$$\int d\mathbf{x}_1d\mathbf{x}_2d\chi d\chi^\dagger e^{-(i/2)(\lambda\mathbf{x}_1^T\mathbf{x}_1+\lambda_b\mathbf{x}_2^T\mathbf{x}_2-\lambda\chi^\dagger\chi)}\times\exp\left\{-\frac{\sigma^2}{8N}\mathrm{Tr}[\hat{Q}^2]+\frac{\sigma^2}{16N}(\chi^\dagger\chi)^2+\frac{\sigma^2}{16N}\chi^\dagger\times(\mathbf{x}_1\otimes\mathbf{x}_1^T+\mathbf{x}_2\otimes\mathbf{x}_2^T)\chi\right\}. \tag{40}$$

In this expression we introduced a positive definite matrix

$$\hat{Q}=\begin{pmatrix}\mathbf{x}_1^T\mathbf{x}_1 & \mathbf{x}_1^T\mathbf{x}_2\\ \mathbf{x}_2^T\mathbf{x}_1 & \mathbf{x}_2^T\mathbf{x}_2\end{pmatrix}.$$

Now we again use the ''decoupling'' of the quartic term in Grassmann variables [the simple Hubbard-Stratonovich transformation Eq. (18)] and then perform the Gaussian integration over anticommuting variables explicitly. The latter yields the determinant factor

$$\det\left[\left(i\lambda-\frac{q\sigma}{\sqrt{N}}\right)\hat{I}_N+\frac{\sigma^2}{2N}(\mathbf{x}_1\otimes\mathbf{x}_1^T+\mathbf{x}_2\otimes\mathbf{x}_2^T)\right]. \tag{41}$$

This factor can be brought to a simpler form

$$\left(i\lambda-\frac{q\sigma}{\sqrt{N}}\right)^{N-2}\det\left[\left(i\lambda-\frac{q\sigma}{\sqrt{N}}\right)\hat{I}_2+\frac{\sigma^2}{2N}\begin{pmatrix}\mathbf{x}_1^T\mathbf{x}_1 & \mathbf{x}_1^T\mathbf{x}_2\\ \mathbf{x}_2^T\mathbf{x}_1 & \mathbf{x}_2^T\mathbf{x}_2\end{pmatrix}\right] \tag{42}$$

by noticing that $(\mathbf{x}_1\otimes\mathbf{x}_1^T+\mathbf{x}_2\otimes\mathbf{x}_2^T)=\hat{X}\hat{X}^T$, where $\hat{X}=(\mathbf{x}_1,\mathbf{x}_2)$ is $N\times2$ rectangular matrix, and using the identity $\det(I_N-\hat{X}\hat{X}^T)=\det(I_2-\hat{X}^T\hat{X})$, then recognizing that $\hat{Q}$ introduced by us above is just 2×2 matrix $\hat{X}^T\hat{X}$. We see that the resulting expression depends on the vectors $\mathbf{x}_{1,2}$ only via the matrix $\hat{Q}$. In recent papers [13] it was shown that the integration over $\mathbf{x}_{1,2}$ under these conditions can be replaced by that over $\hat{Q}$, with an extra factor $\det Q^{(N-3)/2}$ arising in the integration measure. After some straightforward manipulations we arrive at the following integral representation for the required ensemble average:

$$\langle\cdots\rangle_{\hat{H}_2}\propto\int_{-\infty}^{\infty}dq\,e^{-q^2}\left(i\lambda-\frac{q\sigma}{\sqrt{N}}\right)^{N-2}\int_{Q>0}d\hat{Q}\,\det Q^{(N-3)/2}\exp\left\{-\frac{\sigma^2}{8N}\,\mathrm{Tr}[\hat{Q}^2]-\frac{i}{2}\,\mathrm{Tr}\left[\hat{Q}\begin{pmatrix}\lambda & 0\\ 0 & \lambda_b\end{pmatrix}\right]\right\}$$
$$\times\det\left[\left(i\lambda-\frac{q\sigma}{\sqrt{N}}\right)\hat{I}_2+\frac{\sigma^2}{2N}\hat{Q}\right]. \tag{43}$$

Introducing the variables $q_F=-i\lambda+(q\sigma/\sqrt{N})$ and $\hat{Q}_b=(\sigma^2/2N)\hat{Q}$ and shifting the contour of integration in such a way that the integral over q_F goes along the real axis, we can rewrite the above expression as

$$\langle\cdots\rangle_{\hat{H}_2}\propto\int_{-\infty}^{\infty}dq_F\,q_F^{N-2}e^{-(N/\sigma^2)(q_F+i\lambda)^2}\int_{Q_b>0}d\hat{Q}_b\,\det Q_b{}^{(N-3)/2}\det[-q_F\hat{I}_2+\hat{Q}_b]$$
$$\times\exp\left\{-\frac{N}{2\sigma^2}\,\mathrm{Tr}[\hat{Q}_b^2]-\frac{iN}{\sigma^2}\,\mathrm{Tr}\left[\hat{Q}_b\begin{pmatrix}\lambda & 0\\ 0 & \lambda_b\end{pmatrix}\right]\right\}. \tag{44}$$

At the next step we introduce appropriate polar coordinates in the space of matrices $Q_b>0$:

$$\hat{Q}_b=O^T\begin{pmatrix}p_1 & 0\\ 0 & p_2\end{pmatrix}O,\quad d\hat{Q}_b\propto|p_1-p_2|dp_1dp_2d\mathcal{O}, \tag{45}$$

where $p_{1,2}>0$ and O are 2×2 real orthogonal matrices: $\mathcal{O}^T\mathcal{O}=I_2$, with $d\mathcal{O}$ being the corresponding Haar's measure. Explicitly, we can parametrize

$$\mathcal{O}=\begin{pmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{pmatrix}$$

and $dO=d\phi/(2\pi)$. Substituting these expressions into Eq. (44) and after obvious manipulations we get

$$\left\langle\frac{\det^{1/2}(\lambda\hat{I}_N-\hat{H})}{\det^{1/2}(\lambda_b\hat{I}_N-\hat{H})}\right\rangle_{\hat{H}_2}=C_Ne^{(N/\sigma^2)\lambda^2}\int_{-\infty}^{\infty}\frac{dq_F}{q_F^2}e^{-(N/\sigma^2)(q_F^2+2i\lambda q_F-\sigma^2\ln q_F)}\int_0^{\infty}dp_1\int_0^{\infty}dp_2\frac{|p_1-p_2|}{(p_1p_2)^{3/2}}$$
$$\times(q_F-p_1)(q_F-p_2)\mathcal{I}_x(p_1,p_2)e^{-(N/2\sigma^2)[\mathcal{L}(p_1)+\mathcal{L}(p_2)]}, \tag{46}$$

where we reinstated $\lambda_b=\lambda+i\tilde{x}/N$, $\tilde{x}=2\sigma_2^2x$, and

$$\mathcal{L}(p)=p^2+2i\lambda p-\sigma^2\ln p,$$

$$\mathcal{I}_x(p_1,p_2)=\int d\phi\,e^{x(\sigma_2^2/\sigma^2)[(p_1+p_2)-(p_1-p_2)\cos2\phi]}$$

and, as before, C_N stands for the accumulated constant factors. So far all expressions were valid for finite-size matrices. When $N\to\infty$ the integrals in Eq. (46) are evaluated by the saddle-point method. For the bulk of the GOE spectrum $\lambda\leqslant\sqrt{2\sigma^2}$, the relevant saddle points are $p_{1,2}^{sp}=(-i\lambda+\sqrt{2\sigma^2-\lambda^2})/2$ and $q_F^{sp}=(-i\lambda\pm\sqrt{2\sigma^2-\lambda^2})/2$. Again only the choice $q_F^{sp}=(i\lambda+\sqrt{2\sigma^2-\lambda^2})/2$ yields the leading-order contribution, due to presence of the factors $(q_F-p_{1,2})$ in the integrand. Substituting this choice into the integrand in Eq. (46) and evaluating the Gaussian fluctuations around the saddle-point values finally yields

$$\left\langle\frac{\det^{1/2}(\lambda\hat{I}_N-\hat{H})}{\det^{1/2}\left[\left(\lambda+i\frac{\tilde{x}}{N}\right)\hat{I}_N-\hat{H}\right]}\right\rangle\Bigg|_{\substack{x<0\\N\to\infty}}$$
$$=\exp\left\{x[-i\lambda+\sqrt{2\sigma^2-\lambda^2}]\left(\frac{\sigma_2}{\sigma}\right)^2\right\}. \tag{47}$$

It is easy to repeat the calculation for $x>0$ and find that for any real value of x the result can be written as

$$\left\langle\frac{\det^{1/2}(\lambda\hat{I}_N-\hat{H})}{\det^{1/2}\left[\left(\lambda+i\frac{\tilde{x}}{N}\right)\hat{I}_N-\hat{H}\right]}\right\rangle\Bigg|_{N\to\infty}$$
$$=\exp\left\{-i\lambda x\frac{\sigma_2^2}{\sigma^2}-|x|\pi\rho(\lambda)\sigma_2^2\right\}, \tag{48}$$

where $\rho(\lambda)=(1/\pi\sigma^2)\sqrt{2\sigma^2-\lambda^2}$ is the mean eigenvalue density for GOE.

We see that in the large-size limit $N\to\infty$ the expression for the level curvature distribution induced by purely off-diagonal random coupling $\hat{V}$ between the two subsystems is essentially the same Cauchy-Lorentz distribution for both $\beta=2$ and $\beta=1$ cases, up to rescaling of the widths and the mean value with a simple factor 2:

$$\mathcal{P}_{off}(\mathcal{C})=\frac{1}{\pi}\frac{\sigma_2^2\pi\rho(\lambda)}{\left(\mathcal{C}-\lambda\frac{\sigma_2^2}{\sigma^2}\right)^2+[\pi\rho(\lambda)\sigma_2^2]^2}. \tag{49}$$

We give a more detailed discussion of this issue in the next section.

The distribution of curvatures induced by the block-diagonal contributions $\hat{J}_p$ [the term $\mathcal{C}_{1i}$ in Eq. (5)] for real symmetric matrices is quite different from that of complex Hermitian ones. Performing the ensemble average over the Gaussian distribution of $\hat{J}_1$, Eq. (4) and employing the same methods we find the result to be

$$\langle e^{-ix\mathcal{C}_{1i}}\rangle_{\hat{J}_1}=\prod_{k\neq i}\frac{(\lambda_i^{(1)}-\lambda_k^{(1)})^{1/2}}{\left[\lambda_i^{(1)}+\frac{ix_1}{N}-\lambda_k^{(1)}\right]^{1/2}} \tag{50}$$

with $x_1=2\sigma_1^2x$. The averaging over the joint probability density of $(N-1)$ GOE eigenvalues $\lambda_1^{(1)},\ldots,\lambda_{i-1}^{(1)},\lambda_{i+1}^{(1)},\ldots,\lambda_N^{(1)}$ which are different from the eigenvalue $\lambda_i^{(1)}$ whose curvature we address, shows that [7,8]

$$\langle e^{-ix\mathcal{C}_{1i}}\rangle_{\hat{J}_1,\hat{H}_1}\propto e^{-(N/2\sigma^2)\lambda^2}\times\left\langle\frac{|\det(\lambda-H)|\det^{1/2}(\lambda-H)}{\det^{1/2}\left(\lambda+\frac{ix_1}{N}-H\right)}\right\rangle_{N-1}, \tag{51}$$

where H is a $(N-1)\times(N-1)$ GOE matrix.

The averaging of the ratios of determinants in Eq. (51) can be done, *mutatis mutandis*, by the same technique as above. However, the presence of the absolute value of the determinant makes accurate calculation to be quite lengthy, and it will be presented elsewhere, but the result is compact and it is given by [7,8]

$$\langle e^{-ix\mathcal{C}_{1i}}\rangle_{\hat{J}_1,\hat{H}_1}=\pi\rho(\lambda)\sigma_1^2|x|e^{-i\lambda x(\sigma_1^2/\sigma^2)}K_1(|x|\pi\rho(\lambda)\sigma_1^2), \tag{52}$$

with $K_1(z)$ being the MacDonald function of the order one. Such an expression yields, after the Fourier transform, the Zakrzewski-Delande formula for $\beta=1$:

$$\mathcal{P}_{diag}(\mathcal{C})=\frac{1}{2}\frac{\Gamma_d^2}{[(\mathcal{C}-\langle\mathcal{C}\rangle_d)^2+\Gamma_d^2]^{3/2}}, \tag{53}$$

where the mean value $\langle\mathcal{C}\rangle_d=-\lambda(\sigma_1^2/\sigma^2)$ and the characteristic widths $\Gamma_d=\pi\rho(\lambda)\sigma_1^2$.

The curvature distribution, accounting for both the diagonal and the off-diagonal perturbations of two decoupled subsystems, can be found as the convolution of two distributions:

$$\mathcal{P}(\mathcal{C})=\int_{-\infty}^{\infty}d\mathcal{C}_1\mathcal{P}_{diag}(\mathcal{C}_1)\mathcal{P}_{off}(\mathcal{C}-\mathcal{C}_1) \tag{54}$$

and in this way we arrive at the final formula for the real symmetric case. We present it below for the central point of the spectrum $\lambda=0$:

$$\mathcal{P}(\mathcal{C})=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\mathcal{C}_1\frac{\Gamma_d^2}{[\mathcal{C}_1^2+\Gamma_d^2]^{3/2}}\frac{\Gamma_{off}}{(\mathcal{C}-\mathcal{C}_1)^2+\Gamma_{off}^2}. \tag{55}$$

V. NUMERICAL RESULTS AND DISCUSSIONS

In the present section we compare the derived analytical form of the curvature distribution with the results of direct numerical simulations of the ensemble. For our numerical investigations we used a normal random distribution that was adopted from FORTRAN Numerical Recipes [17] and to find the eigenvalues we superseded some subroutines from LAPACK [18]. To avoid the necessity of unfolding the spectra we took into account only levels around the central part of the spectrum. Namely, for a 100×100 matrix, ten middle eigenvalues (20 or more for larger matrices) were considered at each time step, and a curvature value for each eigenvalue was calculated by a second difference equation

$$\lambda_i''(\varepsilon)|_{\varepsilon=0}=\frac{\lambda_i(-2\varepsilon)+16\lambda_i(-\varepsilon)-30\lambda_i(0)+16\lambda_i(\varepsilon)-\lambda_i(2\varepsilon)}{12\varepsilon^2}. \tag{56}$$

The choice of five points instead of the usual three [5] was made to ensure the stability of the results, especially for the GOE case of our system. The empirical choice of $\varepsilon=0.001$ was an outcome of a number of trials; the values it takes may be system specific. We finally remark that using larger matrices, e.g., 400×400 did not improve the quality of plots considerably.

The normalized results of the simulations are presented in Figs. (1) and (2). To compare them with the analytical predictions, for the GUE-like case $\beta=2$ we consider Eq. (32) at $\lambda=0$ and $\sigma=1$. It is also convenient to use the dimensionless curvatures κ obtained from $\mathcal{C}$ by rescaling the latter with the variance of ''level velocity'' as (cf. Ref. [7])

$$\kappa=\mathcal{C}\frac{\Delta}{\pi\left\langle\left(\frac{d\lambda_i^{(p)}}{d\varepsilon}\right)^2\right\rangle}, \tag{57}$$

where $\Delta=\sigma\pi\sqrt{2}/N$ is the mean level spacing of a single subsystem at $\lambda=0$. After some simple calculation, the first-

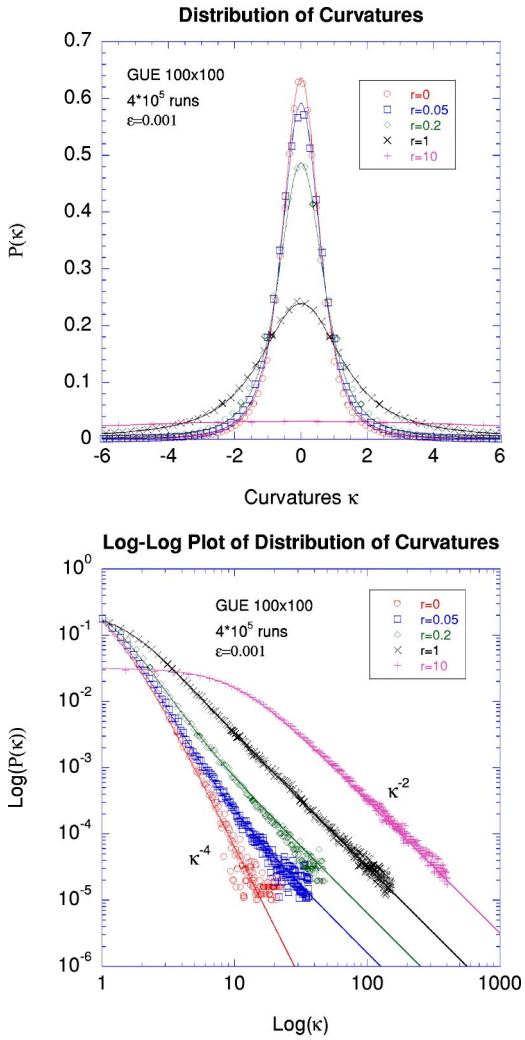

FIG. 1. Normalized GUE curvature distributions for a few selected values of $r=\sigma_2^2/\sigma_1^2$ (intercoupling to intracoupling strengths); bottom plot (log-log scale) shows the tail behavior of the same distribution.

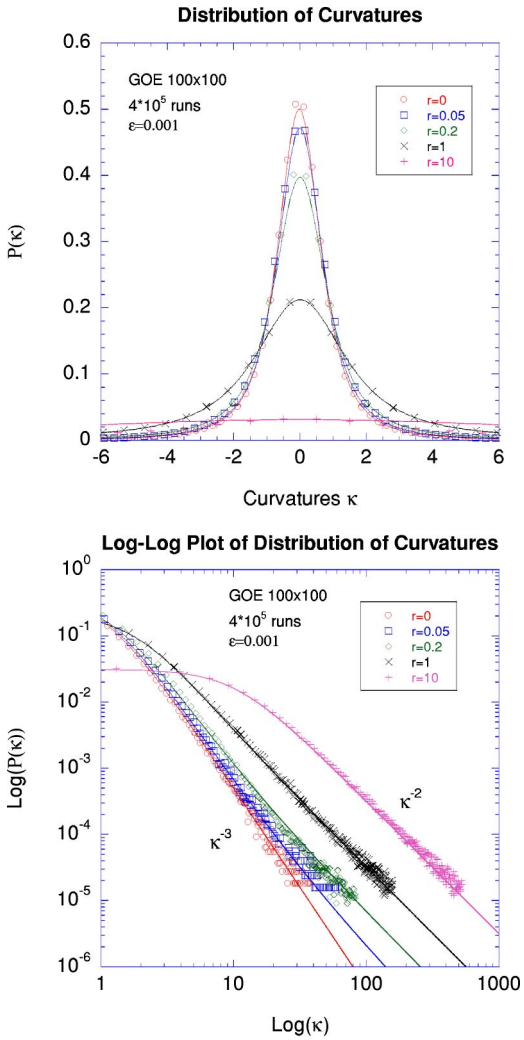

FIG. 2. Normalized GOE curvature distributions for a few selected values of $r=\sigma_2^2/\sigma_1^2$. The solid lines are analytical predictions superimposed over numerical data plots.

order perturbation theory gives $\langle(d\lambda_i^{(p)}/d\varepsilon)^2\rangle=2\sigma^2\sigma_1^2/N$ and is independent of intersubsystem coupling strength σ_2. As a result of the rescaling, the curvature distribution acquires the form

$$\mathcal{P}(\kappa)=\frac{2}{\pi}\left\{\frac{r/2}{[\kappa^2+(1+r)^2]}+\frac{(1+r)^2}{[\kappa^2+(1+r)^2]^2}\right\}, \quad (58)$$

controlled by the only parameter $r=\sigma_2^2/\sigma_1^2$, i.e., the ratio of the intersubsystem to the intrasubsystem coupling strengths. Superimposing the plots of this expression over the appropriately normalized numerical data shows good agreement for all corresponding values of the parameter r, despite the noise in the large curvature tails. For curvatures exceeding the typical value $\kappa \gg r+1$ the distribution shows a power-law tail, the GUE-like behavior κ^{-4} being replaced by the Cauchy-Lorentz one κ^{-2} with the relative growth of the ratio r. For any $r>0$ the most distant tail is always of the Cauchy-Lorentz type, but intermediate GUE-like behavior is clearly seen for $r \ll 1$ when intrasubsystem coupling appreciably exceeds the intersubsystem one. The crossover curvature value between the two regimes of decay is approximately described by the expression

$$\kappa_{cr} \simeq \sqrt{2}\left(\frac{1}{\sqrt{r}}+\sqrt{r}\right), \quad (59)$$

which can be obtained by equating the large-curvature tails originating from the two competing terms in the expression Eq. (58). For curvatures in the interval $1 \ll \kappa \ll \kappa_{cr} \simeq \sqrt{2/r}$, the behavior is GUE-like, changing to a slower Cauchy-Lorentz decay at $\kappa \gg \kappa_{cr} \simeq \sqrt{2/r}$. It is also easy to verify that the maximal value of the distribution $P_{max}=\mathcal{P}(0)$ always decreases with the increasing ratio

$$\mathcal{P}(0)=\frac{1}{\pi}\left[\frac{1}{r+1}+\frac{1}{(r+1)^2}\right] \quad (60)$$

For the case of GOE, a similar rescaling of curvatures at $\lambda=0$ and $\sigma=\sqrt{2}$ leads from Eq. (55) to the expression

$$\mathcal{P}(\kappa)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dx\,\frac{1}{(x^2+1)^{3/2}}\,\frac{r}{r^2+(x-\kappa)^2}. \qquad (61)$$

We plot this distribution superimposed over the numerical data for various values of r. Again, they agree rather well with the numerics and a crossover behavior from GOE-like tail to Cauchy-Lorentz one can be seen clearly for small r, e.g., $r=0.05$.

In fact, one may notice from our plots that decrease in maximal value of the distribution starts to be noticeable at much smaller values of r than the modification of tail behavior at not very large values of κ. This fact qualitatively corroborates with the experimental observations in quartz blocks [9], where noticeable deviations were detected in the center of the distribution, whereas the tails agreed well. Although, our oversimplified model clearly cannot be considered as adequate for describing the actual experimental situation, we nevertheless mention that the choice of $r\simeq 0.2$ allows matching the drop of the peak value with the experimentally observed deviation and produces an overall good agreement with the experimental curve.

As we already noted in the text of the paper, and as clearly seen from the numerical log-log plots, the limiting large-N curvature distribution due to purely off-diagonal (intersubsystem) perturbations turned out to have the same Cauchy-Lorentz form irrespective of the underlying symmetry, for both $\beta=2$ and $\beta=1$. Below we give an alternative, heuristic derivation of this fact, which sheds some light on the origin of such a behavior. The starting point for our analysis is expression Eq. (13) for $\beta=2$ or Eq. (35) for $\beta=1$. Denoting $\lambda_i^{(1)}=\lambda$, as in the text above, we rewrite those formulas as

$$\begin{aligned}\left\langle \prod_{k=1}^{N}\left[\frac{(\lambda-\lambda_{2,k})}{\left(\lambda+\frac{i\tilde{x}}{N}\right)-\lambda_{2,k}}\right]^{\beta/2}\right\rangle_{\hat{H}_2} &= \left\langle \prod_{k=1}^{N}\left[1-\frac{i\tilde{x}}{N}\,\frac{1}{\left(\lambda+\frac{i\tilde{x}}{N}\right)-\lambda_{2,k}}\right]^{\beta/2}\right\rangle_{\hat{H}_2}\\ &= \left\langle \exp\left\{\frac{\beta}{2}\sum_{k=1}^{N}\ln\left[1-\frac{i\tilde{x}}{N}\,\frac{1}{\left(\lambda+\frac{i\tilde{x}}{N}\right)-\lambda_{2,k}}\right]\right\}\right\rangle_{\hat{H}_2}\\ &\approx \left\langle \exp-\frac{\beta}{2}\sum_{k=1}^{N}\frac{i\tilde{x}}{N}\,\frac{1}{\left(\lambda+\frac{i\tilde{x}}{N}\right)-\lambda_{2,k}}\right\rangle_{\hat{H}_2}. \end{aligned} \qquad (62)$$

At the last step we made a plausible assumption that the above in expression the limit of large N can be approximated by expanding the logarithms in the exponential to the first nonvanishing term. Now we introduce an *exact* eigenvalue density for $\hat{H}_2$ as

$$\rho(\mu)=\frac{1}{N}\sum_{k=1}^{N}\delta(\mu-\lambda_{2,k}), \qquad (63)$$

where $\delta(x)$ stands for the Dirac delta function. In terms of this density the ensemble average (62) can be rewritten as

$$\left\langle \exp\left\{-i\frac{\beta}{2}\tilde{x}\int d\mu\rho(\mu)\,\frac{1}{\left(\lambda+\frac{i\tilde{x}}{N}\right)-\mu}\right\}\right\rangle_{\hat{H}_2}. \qquad (64)$$

Now, we use the well-known fact that the exact eigenvalue density for random matrices is *self-averaging*, which means in the limit $N\to\infty$ it converges to a *nonrandom* smooth function, the mean eigenvalue density. The latter function is just given by the Wigner semicircular law $\rho_{sc}(\mu)=(2/\pi\mu_{sc}^2)\sqrt{\mu_{sc}^2-\mu^2}$ for $|\mu|<\mu_{sc}$, where $\mu_{sc}=\sqrt{8\sigma^2}$ for GUE and $\mu_{sc}=\sqrt{2\sigma^2}$ for GOE. All these facts suggest that in the limit of large N the ensemble average in Eq. (64) can be suppressed in favor of replacing the exact density with its semicircular form. Moreover, since $\rho_{sc}(\mu)$ is a smooth function, in the limit of $N\to\infty$ we can use the Sohotsky formula

$$\begin{aligned}\lim_{N\to\infty}\int_{-\mu_{sc}}^{\mu_{sc}}d\mu\rho_{sc}(\mu)\,\frac{1}{\left(\lambda+\frac{i\tilde{x}}{N}\right)-\mu}&\\ =\mathcal{P}\int_{-\mu_{sc}}^{\mu_{sc}}d\mu\rho_{sc}(\mu)\frac{1}{(\lambda-\mu)}-i\,\mathrm{sgn}[\tilde{x}]\pi\rho_{sc}(\lambda),&\end{aligned} \qquad (65)$$

where the first term is understood as a principal value inte-

gral, and sgn stands for the sign function of the argument. In fact, with some effort the integral can be evaluated explicitly:

$$\frac{2}{\pi\mu_{sc}^2}\mathcal{P}\int_{-\mu_{sc}}^{\mu_{sc}} d\mu\,\frac{\sqrt{\mu_{sc}^2-\mu^2}}{(\lambda-\mu)}=2\frac{\lambda}{\mu_{sc}^2}. \tag{66}$$

Collecting all terms we see that

$$\left\langle\left[\frac{\det(\lambda\hat{I}_N-\hat{H})}{\det\left[\left(\lambda+i\frac{\tilde{x}}{N}\right)\hat{I}_N-\hat{H}\right]}\right]^{\beta/2}\right\rangle\Bigg|_{N\to\infty}$$
$$=\exp\left\{-\frac{\beta}{2}\left[i\tilde{x}\frac{2\lambda}{\mu_{sc}^2}+|\tilde{x}|\,\pi\rho_{sc}(\lambda)\right]\right\}, \tag{67}$$

which coincides with the earlier derived expressions in Eqs. (24) and (48).

It is natural to expect that such a derivation can be made mathematically rigorous. However, despite its simplicity and conceptual clarity, such a method cannot be straightforwardly applied to evaluation of the more complicated averages such as those in Eqs. (27) and (51). Indeed, the application of the outlined procedure to Eq. (27) amounts to approximating the extra determinant factor in the large-N limit as

$$\det{}^2(\lambda\hat{I}_N-H_2)\approx\exp\left\{2N\frac{2}{\pi\mu_{sc}^2}\mathcal{P}\int_{-\mu_{sc}}^{\mu_{sc}} d\mu\sqrt{\mu_{sc}^2-\mu^2}\right.$$
$$\left.\times\ln(\lambda-\mu)\right\}=e^{2N\lambda^2/\mu_{sc}^2}, \tag{68}$$

which is indeed a correct expression up to the *leading* order in N in the exponential. It serves to cancel the extra factor $e^{-N\lambda^2/4\sigma^2}$ in front of the ensemble average in Eq. (27). However, it is easy to understand that to arrive at the correct expression Eq. (29) one needs to take into account *subleading* terms—those of the order of unity in the exponential. This goal goes beyond the simple use of self-averaging, and requires a much more detailed treatment. It is not clear at the moment how to implement such a treatment in the present heuristic scheme. That is why the supersymmetrization method, which yields fully controllable results in all cases should be in general preferred.

In conclusion, we have derived exact expressions for the distribution of level curvatures in a model describing a mixing of two independent spectra by a generic perturbation. Although the model is too simple to describe actual experimental situation in systems with partially broken symmetries, some features of the behavior of our curvature distribution may play a role of useful analogy helping to understand the deviations in experimentally measured level curvature distribution of the acoustic resonances of quartz blocks [9]. Indeed, the maximum value of the latter distribution was found to be considerably lower than that predicted for pure GOE case, whereas the tail shows good $\mathcal{C}^{-3}$ decay. This agrees qualitatively with our observation that the peak value of the curvature distribution might be more sensitive to remnant symmetries than the power law tail behavior.

In fact, an ideal experimental realization of our model may be the system of two superconduction microwave billiards coupled by an antenna in a variable way [19]. Although, in real experiments of this type the coupling was changed in large discrete increments, it is in principle possible to change it in a much more controllable way, and to study level dynamics induced by such a coupling. We hope that our results may stimulate experiments of this sort.

ACKNOWLEDGMENTS

We would like to thank J. Zakrzewski and M. Oxborrow for sending us copies of their articles [6,10] and EPSRC for the financial support.

[1] F. Haake, *Quantum Signatures of Chaos*, 2nd ed. (Springer-Verlag, Berlin, 2000).
[2] T. Guhr, A. Müller-Groeling, and H.A. Weidenmüller, Phys. Rep. **299**, 189 (1998).
[3] F.J. Dyson, J. Math. Phys. **3**, 140 (1962).
[4] M.L. Mehta, *Random Matrices*, 2nd ed. (Academic Press, San Diego, 1991).
[5] P. Gaspard, S.A. Rice, and K. Nakamura, Phys. Rev. Lett. **63**, 930 (1989); P. Gaspard, S.A. Rice, H.J. Mikeska, and K. Nakamura, Phys. Rev. A **42**, 4015 (1990).
[6] J. Zakrzewski and D. Delande, Phys. Rev. E **47**, 1650 (1993).
[7] F. von Oppen, Phys. Rev. E **73**, 798 (1994); **51**, 2647 (1995).
[8] Y.V. Fyodorov and H.-J. Sommers, Phys. Rev. E **51**, R2719 (1995); Z. Phys. B: Condens. Matter **99**, 123 (1995).
[9] P. Bertelsen, C. Ellegaard, T. Guhr, M. Oxborrow, and K. Schaadt, Phys. Rev. Lett. **83**, 2171 (1999).
[10] M. Oxborrow and C. Ellegaard (unpublished).
[11] M.S. Hussein, C.P. Malta, M.P. Pato, and A.P.B. Tufaile, Phys. Rev. E **65**, 057203 (2002).
[12] A.A. El-Hady, A.Y. Abul-Magd, and M.H. Simbel, J. Phys. A **35**, 2361 (2002).
[13] Y.V. Fyodorov, Nucl. Phys. B **621**, 643 (2002); Y.V. Fyodorov and E. Strahov, *ibid.* **630**, 453 (2002).
[14] M.S. Hussein and M.P. Pato, Phys. Rev. Lett. **84**, 3783 (2000), and references therein.
[15] Y.V. Fyodorov and E. Strahov, J. Phys. A **36**, 3203 (2003).
[16] K.B. Efetov, *Supersymmetry in Disorder and Chaos* (Cambridge University Press, Cambridge, 1997).
[17] See http://www.nr.com
[18] See http://www.netlib.org/lapack
[19] H. Alt *et al.*, Phys. Rev. Lett. **81**, 4847 (1998).